Matrix Rank and Span

Recall Theorem 1: If A is a $m \times n$ matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if $\text{rank}(A) = m$.

Theorem 2: Let A be a $m \times n$ matrix with column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then the following three statements are equivalent. (they are either all TRUE statements or all FALSE statements).

1. $\text{rank}(A) = m$
2. The linear system $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$.
3. $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \mathbb{R}^m$

(1) $\Leftrightarrow$ (2) Theorem 1.

(2) $\Leftrightarrow$ $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$.

$\Leftrightarrow$ $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}$ has a sol$^{\underline{n}}$ for all $\vec{b}$

$\Leftrightarrow$ any $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$

$\Leftrightarrow$ $\text{span}(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n) = \mathbb{R}^m \Leftrightarrow$ (3).

Example 5: Use theorem 2 to justify the following equality.

$$\operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right) = \mathbb{R}^3 \tag{1}$$

$A = \begin{bmatrix}1 & 2 & 4 & 7\\0 & 3 & 5 & 8\\0 & 0 & 6 & 9\end{bmatrix}$. Since A is in row echelon form, $\operatorname{rank}(A)=3$

Since $\operatorname{rank}(A)=3$, equation (2) holds by theorem 2.

Example 6: Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}-1\\-1\\-2\end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix}2\\-1\\1\end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix}3\\4\\8\end{bmatrix} \tag{2}$$

Show that $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \mathbb{R}^3$.

$$A = \begin{bmatrix}1 & -1 & 2 & 3\\2 & -1 & -1 & 4\\3 & -2 & 1 & 8\end{bmatrix} \underset{R_2 := R_2 - 2R_1}{\overset{R_3 := R_3 - 3R_1}{\sim}} \begin{bmatrix}1 & -1 & 2 & 3\\0 & 1 & -5 & -2\\0 & 1 & -5 & -1\end{bmatrix} \overset{R_3 := R_3 - R_2}{\sim}$$

$$\sim \begin{bmatrix}1 & -1 & 2 & 3\\0 & 1 & -5 & -2\\0 & 0 & 0 & 1\end{bmatrix} \qquad \operatorname{rank}(A)=3$$

Since $\operatorname{rank}(A)=3$, $\operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \mathbb{R}^3$ by theorem 2.